

Technology-Enhanced Mathematics Instruction:
Effects of Visualization on Student Understanding of Trigonometry

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Abstract

The purpose of this study was to investigate the effectiveness of technology for visualization upon student understanding of the trigonometric concepts of radian, reference angle, and the Unit Circle. The participants in this mixed-methods study were two groups of high school precalculus students who were beginning a unit on trigonometry. One group was designated as the comparison group and received traditional instruction from an experienced teacher who had taught high school mathematics for over ten years. The other group was the visualization group and received its technology-enhanced instruction from a preservice teacher who was given this assignment as part of her student teaching experience. This researcher, who was assigned as her supervisor during the student teaching experience, provided pedagogical and technical support during the six weeks of the study. The study took place under the auspices of a federally-funded grant to investigate the effectiveness of technology in mathematics and science instruction. The main finding was that students who received the technology-enhanced whole-class instruction focusing on visualization demonstrated a higher level of understanding of radians, reference angles and the Unit Circle. More specifically, use of the technology, including animation and dynamic image manipulation, was found to have a positive effect upon student learning. Of additional interest was the finding, based upon their performance on classroom written assessments, that students in the visualization group exhibited no loss of procedural competence. These findings have implications for teacher educators who prepare future mathematics teachers and also for school systems as they integrate technology into mathematics classrooms.

Mathematics educators have long recognized the importance of multiple representations in helping students develop an understanding of mathematical concepts. In recent years, advances in technology have given teachers and students more options for producing such representations quickly and accurately. At every level of secondary mathematics including the study of trigonometry, representation plays an important role. Very little research exists on the effectiveness of technology use for visualization upon students' understanding of trigonometry concepts, and it is this area upon which the following research study concentrated.

The National Council of Teachers of Mathematics (NCTM) lists representation as one of the five mathematical process standards in Principles and Standards for School Mathematics (2000), a noteworthy departure from its earlier position in which representation was included as part of the communication standard (1989). Since representation plays a role in communication and as a tool for thinking (Greeno & Hall, 1997), it is now given even greater attention than before.

The Representation Standard states that instructional programs from prekindergarten through grade twelve should enable all students to:

- (1) create and use representation to organize, record, and communicate mathematical ideas;
- (2) select, apply, and translate among mathematical representations to solve problems;
- (3) use representations to model and interpret physical, social, and mathematical phenomena.

In secondary mathematics and especially in relation to the study of functions, students learn many ways in which a function may be represented, but the emphasis tends to be on the

“big three:” algebraic, numerical, and graphical (Nemirovsky, Tierney, and Wright, 1998). Greeno and Hall (1997) made several observations about the importance of representations:

- (1) Representations are powerful tools for thinking.
- (2) Understanding and mathematical concepts and procedures are enhanced when student can transfer understanding among different representations.
- (3) Representations give learners useful tools for building understanding, communicating information, and demonstrating reasoning.

The NCTM also formulated a Technology Principle which states “Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning” (2000, p. 24). The Technology Principle suggests that technology can support effective mathematics teaching, given the teacher’s evolving role in today’s technology-equipped classroom. The NCTM believes that teachers should take advantage of technology such as spreadsheets, dynamic geometry software, and computer microworlds and use it appropriately as a tool to improve instruction. The NCTM’s Professional Development Focus of the Year for 2006-2007 was “Learning through Representation,” and their web site states “Teachers should also encourage the use of multiple representations and technology-assisted representations” (NCTM, n.d.).

Existing research whose focus was on the use of technology in the teaching of trigonometry is exceedingly limited. Since the study of trigonometry is an important prerequisite for a course in calculus, this is a subject that needs to be investigated to determine the most effective instructional approaches. Furthermore, with the number of calculus students expected to rise dramatically in the next decade (College Board, 2005), the demand for trigonometry will continue to increase.

Objectives of the Research Study

Recognizing that representation is important in teaching and learning mathematics and acknowledging that appropriate use of technology can facilitate the integration of representations in classroom instruction, this research study investigated the effectiveness of technology use for visualization in a whole-class setting on student understanding of several important concepts in trigonometry. Specifically this study sought to answer the following research questions:

- (1) How does the use of visualization and animation with technology in a whole-class instructional setting affect students’ learning of trigonometry concepts of radian, reference angle, and the Unit Circle?
- (2) What misconceptions do students have in connection with important topics in trigonometry?

Methods

This study employed a mixed method approach (Johnson & Onwuegbuzie, 2004) consisting of a modified experimental design as well as a qualitative component. Johnson and Onwuegbuzie define mixed methods research as “the class of research where the researcher mixes or combines quantitative and qualitative research techniques, methods, approaches, concepts or language into a single study” (2004, p. 17). They note that mixed methods research uses “multiple approaches in answering research questions, rather than restricting or constraining researchers’ choices” (2004, p. 17). Further they state that this form of research has certain advantages over either a purely quantitative or a purely qualitative method:

It is an expansive and creative form of research, not a limiting form of research. It is inclusive, pluralistic, and complementary, and it suggests that researchers take an eclectic approach to method selection and the thinking about and conduct of research. What is most important is the research question—research methods should follow research questions in a way that offers the best chance to obtain useful answers (2004, p. 17-18).

I felt that both quantitative and qualitative approaches offered beneficial attributes in a study of this type and that the exclusion of either type of methodology would reduce the opportunity to investigate the effectiveness of the technology treatment. A purely quantitative approach would allow me to compare groups and subgroups of students according to scores earned on written evaluations such as quizzes and tests. This straightforward approach would provide the mechanism to test hypotheses regarding the effectiveness of the treatment, although it also limited the manner in which I could make comparisons between groups.

A qualitative approach, on the other hand, would afford me the opportunity to collect and analyze a wide variety of data to look for themes and make comparisons between groups based upon my analysis of the data. This approach also would provide a setting in which conceptions and misconceptions regarding students' understanding of trigonometry could surface unencumbered. With this mixed-method research design in mind, data from several sources was collected for analysis, including two types of student interviews, extensive classroom observations, and student work samples in the form of quizzes and tests.

Participants

The participants in this study were two classes of high school honors precalculus students in a suburban middle Atlantic state who were beginning a unit on trigonometry. These students were selected out of twenty-five classes that were being taught by a group of student teachers at the state university serving this region. Several factors influenced the decision to work with these two classes. First, to minimize the teacher effect, we preferred to work with two classes of students who were being taught at the same level by the same teacher, rather than confound the study by comparing students in different classes or taught by two different teachers. Second, the research team wanted to work with students who were enrolled in a course above the level of algebra and geometry in order to study a level of mathematics that had not been the subject of extensive prior research. Finally, because this study was part of a federally funded grant, it was necessary to include preservice teachers in order to satisfy the grant requirements.

One group ($n = 13$) was identified as the comparison group and received traditional instruction from an experienced teacher, Mr. Brownell, who had taught high school mathematics for over ten years. The other group ($n = 13$) was designated as the visualization group and received its technology-enhanced instruction from a preservice teacher, Stella, who was given this assignment as part of her student teaching experience. Mr. Brownell was Stella's cooperating instructor and charged with helping her to make the transition into full-time teaching during her student teaching semester. In this instance, however, for the purposes of the research study, Mr. Brownell continued to teach one section of precalculus while Stella taught the other. The two sections were comparable in background and achievement levels according to baseline data collected during the month of the school year that preceded the study.

Mr. Brownell had been teaching high school mathematics at Montibello High School since it opened eight years earlier, and he had taught for several years in another state prior to that. He had been teaching precalculus for the entire time that he had been teaching at this school, and he also served as the school's head football coach. Stella was in her final year of a five-year program of study leading to a bachelor's degree in mathematics and a master's degree

in teaching, and she was considered one of the most capable members of her cohort of teachers. The university conducting this study is a highly selective state university with an excellent reputation for teacher preparation, and its school of education includes a center whose focus is on the use of technology in teacher education.

I was working as a graduate fellow in the center for technology education and assigned as Stella's supervisor during the student teaching experience as part of my graduate responsibilities. This study took place under the auspices of a federally-funded grant to investigate the effectiveness of technology use in mathematics and science instruction. As part of this larger study I was assisted by two others who were members of the project team. One person was a graduate student in mathematics education at the supervising university, and the other was the grant principal investigator and an associate professor in the school of education at the same university.

Experimental Treatment

The quasi-experimental design of the study called for the two groups of students to receive different types of classroom instruction for a period of six weeks, during which data would be collected to determine the effectiveness of the treatment afforded the experimental group. The control group was taught by Mr. Brownell and received traditional instruction utilizing a white board, an overhead projector, and the graphing calculator. These tools are standard equipment in most high school mathematics classrooms, and these were also the tools that Mr. Brownell was accustomed to using with his mathematics classes.

The experimental group was known as the visualization group and was taught by Stella using a SMART Board, an LCD projector, a laptop computer, and software such as PowerPoint, the Geometer's Sketchpad, TI-SmartView, and a customized Flash animation that illustrated the concept of radian. Stella also had access to the Internet and a subscription to Explore Learning, a website that offered interactive simulations for her to use in whole class instruction. Students had access to their graphing calculators during class and on their homework assignments.

The pacing and sequencing of the content was determined by Mr. Brownell, who was also responsible for the creation of all classroom assessments. Typically Mr. Brownell taught a lesson first while Stella and I observed. Because the classes taught by the two teachers usually met on alternate days, Stella was responsible for teaching the same lesson the following day, which afforded us an opportunity to discuss and plan her lesson jointly in most cases. Stella's lessons contained the same content presented in a more visual manner whenever possible, taking advantage of the technology she had at her disposal. During the six weeks of the study I was able to provide pedagogical and technical support along with the other members of the research team.

School-initiated schedule changes occurred during the study which made it impossible to adhere to the original design with regard to the order in which classes were taught. These changes required Stella to teach her class prior to the corresponding class taught by Mr. Brownell for approximately one-third of the study. Therefore Stella and I did not have an opportunity to observe Mr. Brownell's class first and then plan her lesson; instead Stella and I had to develop a lesson independent of Mr. Brownell based only on our knowledge of the topics that were to be covered on a given day.

Data Collection

During the study several types of data were collected and subsequently analyzed. These included one classroom test and two quizzes which were administered to both classes, two sets of student interviews conducted at the middle and at the end of the study, daily observations of classroom instruction by both teachers as well as audio recordings of each class.

The first quiz was created by Mr. Brownell and contained mostly items of a procedural nature, making it difficult to assess students' level of understanding of the trigonometric concepts which were the focus of this study. Consequently we asked Mr. Brownell if he would allow Stella, in collaboration with the members of the project team, to create the next quiz, subject to his approval. Mr. Brownell readily agreed and subsequently accepted the quiz we produced without modification. Mr. Brownell designed the test that was administered several days after the second quiz and included many items that were similar to those which had appeared on either the first or second quiz. These two quizzes and one test constituted the bulk of the artifacts of student work that were analyzed as part of this study.

The research team members conducted one-on-one structured interviews with the participants at two points during the study, each interview lasting approximately 15 to 20 minutes. These interviews used a standardized protocol to ensure that every session was similar in format, regardless of who conducted the interview. The first set of interview questions focused on items from the first two quizzes on which students were asked to explain their thinking as they solved the problems. Much of the second interview was a task-based interview that included tasks related to the final two weeks of instruction which are not included in this report. Only one question from the second interview is included in this set of data.

Level of Understanding	Points Deducted	Points Earned	Description
High	0	5	Complete and correct answer with indication of correct understanding of concept. Answer accurate and presented in form specified.
	1	4	Almost complete, nearly correct except for minor computational error. No indication of any misunderstanding of concept. Answer may be correct except for format.
Medium	2	3	Partially correct with indication of some misunderstanding of concept. May also include computational errors.
	3	2	Partially correct with significant lack of understanding evident. Likely to include computational errors as well.
Low	4	1	Some work shown but little evidence that student understands concept or is making progress toward the desired result.
	5	0	Student did not answer the question at all, leaving the section blank or offered only a minimal effort at solving the problem.

Figure 1. Expert grader item scoring rubric

Data Analysis

The student work samples were analyzed by a team of three expert graders who assigned scores for each item on the two quizzes and one test which made up the written classroom assessments. All three graders had taught high school mathematics for a period of time ranging from three to twenty-eight years, and all also had experience teaching trigonometry. This team of graders independently scored every item using an item scoring rubric that I had developed (see Figure 1). The scores were then compared and discussed in a face-to-face meeting until a consensus was reached for each item completed by each student in both classes. Averages of the scores on each item by group were then compiled so that a direct class-to-class comparison could be made. This comparison constituted the quantitative portion of the study and served as a first step in assessing differences between the visualization and comparison groups.

The second important data source was the set of interviews conducted by members of the project team with students at two points during the study. The two sets of interviews were transcribed and systematically analyzed for trends and differences between groups. Much of the interview analysis focused on items that were identified based upon mean score differences between groups that showed up in the item analysis, although every piece of student work was examined thoroughly. In cases where mean score differences were small or nonexistent, student work samples showed differences in strategies used by students in the two groups such as the use of sketches or diagrams.

Audio recordings of classroom instruction along with observer field notes were also reviewed throughout the data analysis phase of the study to document differences between the instructional approaches used by the two teachers. This focused on classroom episodes in which the visualization group used technology resources to produce visual images, particularly those involving dynamic image manipulation, and the corresponding presentations used in the comparison group. These analyses of interview transcripts, observation field notes, and audio recordings of lessons represented the qualitative component of the study.

Results

Items from the quizzes and test were organized according to topic which generated three sets: those dealing with radians, those whose focus was reference angles, and those dealing primarily with the Unit Circle. Mean scores by item and by group were compared within these three categories to determine where sizeable differences occurred. Once differences were noted, the interviews and classroom observations were reviewed for additional evidence to substantiate the differences observed in item scores. Together the item scores, interview data, and field notes from classroom observations led to several assertions about differences between groups based upon their modes of instruction.

The main finding was that the students who received the technology-enhanced whole-class instruction focusing on visualization demonstrated a higher level of understanding of radians, reference angles and the Unit Circle than students who had received the traditional instruction. More specifically, use of the technology including animation and dynamic image manipulation was found to have a positive effect upon student learning. These findings are supported by both the quantitative data and the qualitative data collected during the course of this study. The following sections of this paper detail some of the significant findings and provide supporting data and examples of student work and excerpts of interview transcripts that substantiate the findings.

Radians

One of the instructional objectives dealing with radian was for the student to understand what a radian was as well as be able to work with angles given in radians. We addressed both facets of this objective to determine the student's conceptual understanding of radian as well as his procedural ability to operate with radians. On two of the written assessments (Quiz 1 and Test 1) the following item appeared as the first question: "What is a radian?" This same question was also posed in both the first and second interview, giving us a number of sources of data on the same topic. Table 1 shows how the mean scores by class compared on this item on the two written assessments.

Table 1:
Scores earned by students on Q1.1 and T1.1

	Quiz 1 (Item 1)	Test 1 (Item 1)
Group A (comparison)	2.18	2.21
Group B (visualization)	2.25	4.13
Difference (B-A)	0.07	1.92

Although the two group means were nearly identical on the first quiz, the scores on Test 1 differed by almost two full points on the five point scale in favor of the visualization group. The magnitude of this difference prompted us to examine the students' written work closely to see how they had answered the question and whether there were any similarities or differences within and between groups.

1. What is a radian?

$$\text{arc length} = \text{radius}$$

$$\theta = \frac{s}{r}$$

Figure 2. Lauren's response to Test 1.1

The answer given by Lauren (Figure 2) is similar to the answer provided by nearly half of the students in Mr. Brownell's class who made some reference to a formula relating angle, arc length, and radius. Lauren's answer was better than most, because she made a connection between the length of the arc and the length of the radius, but she did rely on the formula like many of her classmates.

1. What is a radian?

measure of the arc length

$$1 \text{ rad} = \text{arc length}$$

→ 1 rad is equal to the measure of its arc.




Figure 3. Megan's answer to Test 1.1

Megan's response (Figure 3) was representative of those given by most of the members of Stella's class in which the relationship between arc length and radius length was stated verbally rather than as a formula. In addition her response included a diagram depicting a radian, complete with both radii and the included arc labeled "r." Her diagram closely resembles the image that appeared when Stella used her Flash animation to introduce the concept of radian to her class (see Figure 4), and Megan also states the relationship between arc length and radius of the circle verbally.

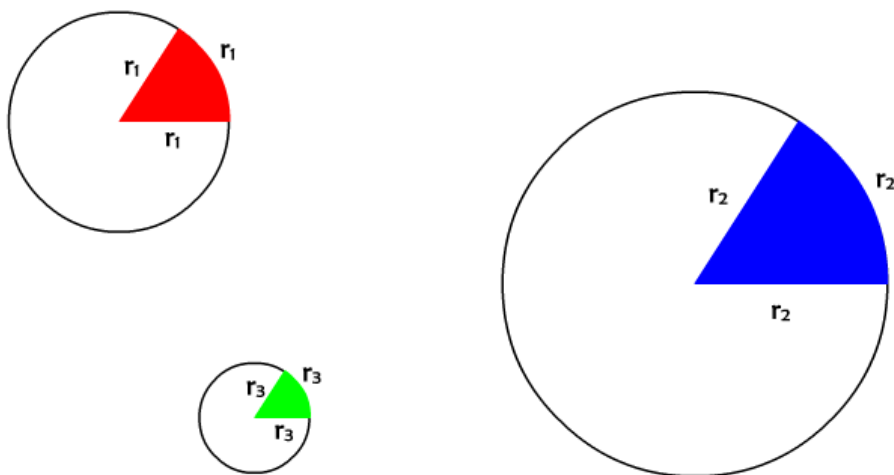


Figure 4. Flash animation image used by Stella

Mr. Brownell, on the other hand, had introduced radian to his class by giving the formal definition along with a hand-drawn static diagram like the one shown in Figure 5.

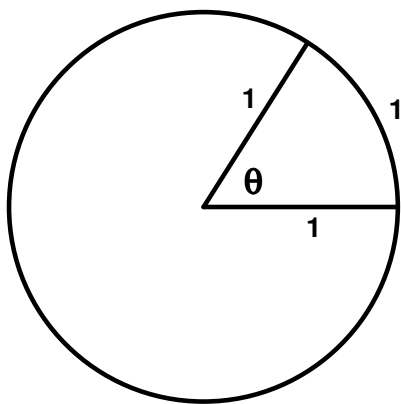


Figure 5. Mr. Brownell's drawing of a radian

Students were told that a radian was “an angle subtended by an arc whose length was equal to the length of the radius of a circle.” Mr. Brownell's diagram showed a circle of radius 1 unit, not a radian requirement but clearly a precursor of the Unit Circle that students would be studying in the near future. In addition, Mr. Brownell introduced students to the formula $\theta = \frac{s}{r}$ relating arc length, central angle measure, and radius length.

Results of the interviews provided additional evidence of the difference between the two groups with regard to their understanding of the concept of radian. Tables 2 and 3 contrast the responses between students in the two groups.

Table 2:
Responses of Mr. Brownell's students given during Interview 1.

Student	Response
Danielle	It's the arc length equals . . . the little circle thing equals s divided by r
Brittany	The arc length over degrees. I think it's arc length over time because it's s over t .
Kevin	That is a unit of measure for a circle and it is arc length over the radius.
Ben	The measure of the angle equals arc length divided by the radius.
Molly	It's arc length divided by the radius.
Austin	It's a way to measure arc length and on a Unit Circle it's usually—well, the estimation he gave us was 57.3 degrees. It's a way to measure angles in circles.
Thomas	The measure of an angle in radians equals the arc length divided by the radius of the circle.
Scott	It's the arc length divided by the radius. One radian, I think, is 57.3 degrees.

Table 3:
Responses of Stella's students given during Interview 1.

Student	Response
Melissa	The two radii form an arc, and the arc formed by the two radii is what the radian is.
Brad	We were using the Geometer's Sketchpad and she took the radius—I remember the arc equal to the radius sweeping out, and she showed us the radius sweep out . . . and everyone said, "that's about 60 degrees."
Glenda	. . . it's a way of measuring an angle and it's around 60 degrees. The arc length is the same as the radius.
Rainey	It's the central angle where the radius and the arc length are equal.
Samantha	. . . a radian is a unit of measure like on the Unit Circle with a hypotenuse of one and the measure has a radius of one and then the radian would be the arc connecting those two.

Students in the comparison group routinely equated a radian to a unit of measure that was related to arc length and radius according to the formula $\theta = \frac{s}{r}$. Two students noted that its degree equivalent was 57.3 degrees because "that's what he told us," whereas Stella's students had the more approximate angle value of "about 60 degrees"

based upon their estimate that slightly more than six radians fit in a circle. Note also how Brad indicated that there was movement associated with his concept of radian. Several other students in the visualization group also used language in the interview that denoted motion or animation similar to what they had seen in class. The differences between classes are consistent with the manner in which the image of a radian was presented: a static version by Mr. Brownell versus an animated version by Stella.

Another quiz and test item that dealt with the concept of radian was the question, “How many radians are in a circle? Explain how you know.” When the two groups were compared, scores on this item were very similar, but an examination of work samples and interview transcripts revealed a noteworthy difference between groups. Students in the comparison group showed a preference for the formula as part of their explanation, while visualization group students opted to explain by making references to the manner in which radians “fit” in a circle during the radian demonstration Stella used with the class on two occasions. Visualization group students were also more likely to use the 60 degree approximate value of a radian, while comparison group students cited the more exact value of 57.3 degrees both in the work samples and in the interviews.

The interview conducted by Dr. Garfield, one of the research team members, with Austin, a member of the comparison group, is indicative of the difficulty students in that group had making a connection between the definition of a radian and the number of radians in a circle. Dr. Garfield asked Austin to explain how he knew that there were two pi radians in a circle.

Austin: Because the circumference is two pi times the radius, and the radius in the Unit Circle is one. If you multiply it out, it’s two pi.

Dr. Garfield: What if it’s not a Unit Circle? How many radians would be in that circle?

Austin: Two pi?

Dr. Garfield: And why is that?

Austin: I’m not very sure why, but

Dr. Garfield: Well, why do you think?

Austin: Because that’s what they teach me.

Dr. Garfield: Can you tell me anything else about a radian?

Austin: It’s measured in revolutions sometimes, isn’t it?

Dr. Garfield: What do you mean “it’s measured in revolutions?”

Austin: Like added up, radians will result in revolutions of a circle.

Dr. Garfield: Do you know any other way to explain why there are two pi radians in a circle? Do you know where the 57.3 came from?

Austin: I remember we had gone over it, but I don’t remember.

These differences between groups on the items scored by the team of expert graders, in the work samples, and during the interviews led to the formulation of the following assertion:

Students in the treatment group demonstrate a better understanding of the concept of radian than students in the comparison group.

Reference Angles

Four items that appeared on Quiz 2 and Test 1 dealt with the concept of reference angles:

- Quiz 2.12 What is the size of the reference angle for an angle $\theta = 4.8$ radians?
- Test 1.21 What is the size of the reference angle for an angle $\theta = 4$ radians?
- Quiz 2.8 Name all angles between 0 and 2π with a reference angle of $\frac{3\pi}{8}$ radians.
- Test 1.17 Name all angles between 0 and 2π with a reference angle of $\frac{2\pi}{5}$ radians.

Table 4 shows how students in the two groups performed on each of the four items. The difference between groups on each item favors the visualization group by an average margin of 0.71 points on the five point scale. We examined the student work samples on the four items and also studied interview transcripts to see if there were any differences between groups.

Table 4:
Scores by group on reference angle items

Item	Group A (comparison)	Group B (visualization)	Difference (B-A)
Quiz 2.12	2.45	3.72	1.27
Test 1.21	2.97	3.74	0.77
Quiz 2.8	2.94	3.21	0.27
Test 1.17	3.54	4.10	0.56
Mean	2.98	3.69	0.71

The comparison of work samples led to an interesting difference between groups when diagrams created by the students were examined closely. The first set of diagrams are those created by students in the visualization group (Figure 6). All of them have common characteristics: the use of x- and y-axes and drawings of four reference angles, one per quadrant. These drawings are strikingly similar to the dynamic image used by Stella when she taught her lessons on reference angles (Figure 7). Her dynamic Geometer's Sketchpad illustration included colorful reference angles in each quadrant that could be manipulated together to show how the size of the reference angle in each quadrant was related to the size of the reference angle in quadrant one.

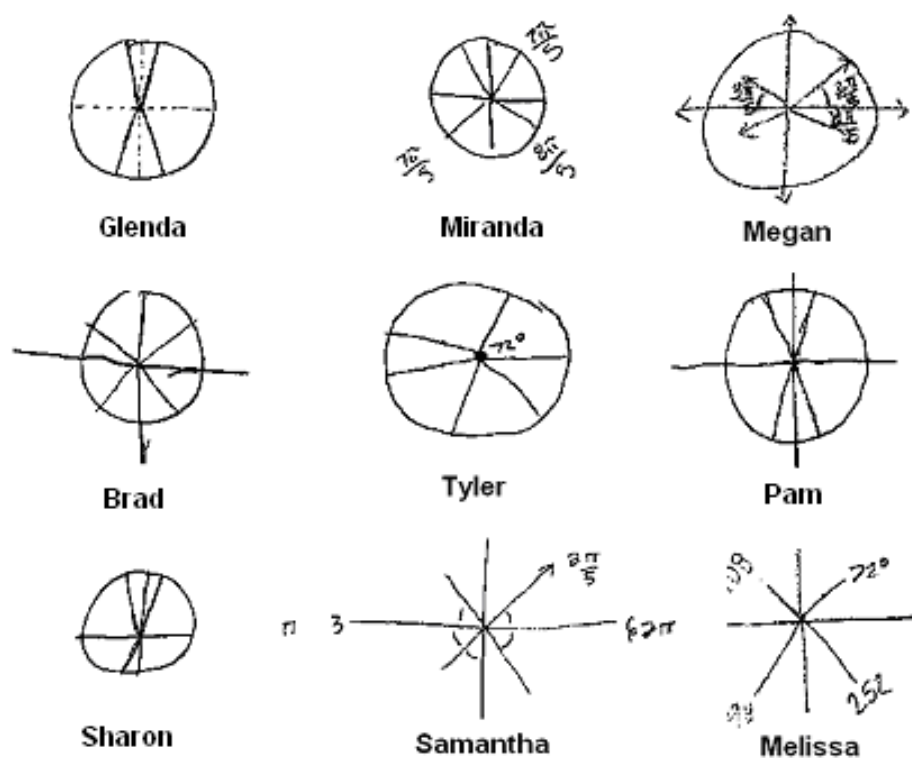


Figure 6. Diagrams created by visualization group students on reference angle question

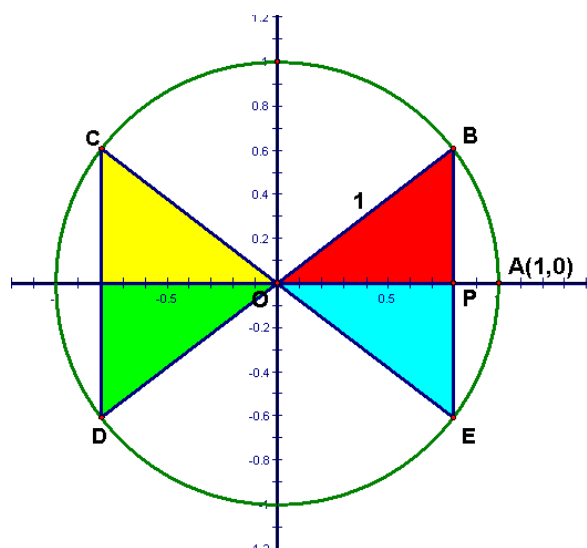


Figure 7. Reference angle image used by Stella with visualization group

Fewer students in the comparison group elected to draw diagrams to aid them in answering the questions about reference angles, and they explained in their interviews that they did not feel it necessary to do so. Those few who did draw diagrams produced the images shown in Figure 8. A comparison between images created by comparison group students and those created by the visualization group shows a clear difference between groups and helps to explain how the students in the visualization group were able to grasp the concept of reference angle more clearly and perform better on the quiz and test items.

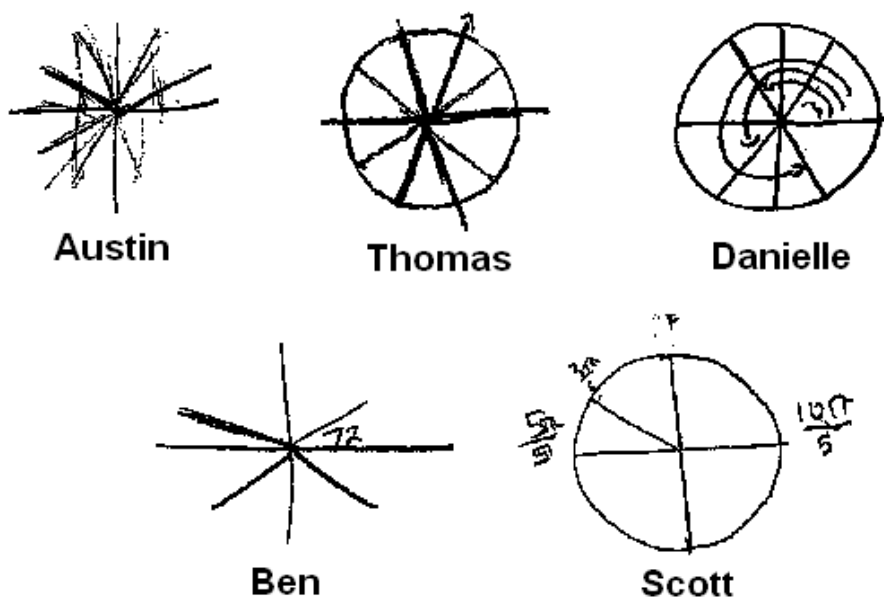


Figure 8. Diagrams created by comparison group students on reference angle question

Based on the item analyses and results of interviews we were able to conclude that a difference did exist between the two groups, and this led to the formulation of a second assertion:

Students in the treatment group demonstrate a better understanding of the concept of reference angle than students in the comparison group.

The Unit Circle

The concept of the Unit Circle is one that presents opportunities and obstacles for students. On the one hand it is a collection of common angles and ordered pairs that may seem to an unsuspecting student like an imposing and impossible set of data to memorize. There are angles measured in both degrees and radians, and there are 16 sets of ordered pairs to learn, one set for each angle on the Unit Circle. Faced with this task, the assignment may appear daunting to the average student.

On the other hand, the Unit Circle may serve as a tool that helps a student easily organize and recall trigonometric values for common angles of any size, whether given in radians or degrees. More importantly, the Unit Circle serves as a bridge between the right triangle trigonometry of the student's past and the graphs of trigonometric functions which lie ahead. A student who understands how the Unit Circle is constructed and appreciates its symmetrical attributes will be able to determine values of angles, coordinates, and trigonometric functions without difficulty, and this will provide the foundational knowledge necessary to develop a deep understanding of the connection between the SOH-CAH-TOA trigonometry he learned in geometry and the graphs of the circular trigonometric functions.

Knowledge of circular functions enables a student to study, analyze, and model applications of a periodic nature, thus opening a new chapter in his study of mathematics. Trigonometric functions can be used to model many real world phenomena such as the motion of the tides, the movement of a pendulum, and the motion of sound waves. For

the calculus-bound student, knowledge of trigonometric functions is a prerequisite, since many integrals and derivatives require the use of trigonometric identities.

For a student to be able successfully to continue his study of trigonometry and the mathematics that follows, he must know the characteristics and applications of the Unit Circle. Students in a precalculus course are expected to learn this and much more about trigonometry, and the same is true of the students in this study. This course is the final course that many of the students take before calculus, whether they take calculus at the high school level or at a community college or four-year university. Students who are sophomores and juniors as they take this course may move on to Calculus AB or Calculus BC the following year, two courses for which they can earn advanced placement college credit. Students in this course have an opportunity to move on to the calculus, something which over 90 percent of high school students nationwide are not able to do, which makes this a very select group of students.

Based on the results of the pre-assessment, students in the two groups began their study of the Unit Circle at the same level with no prior knowledge of the topic. The written assessments and interviews that were subsequently conducted during this study provide strong evidence that a noticeable difference between groups did develop over time. This evidence leads to the formulation of the third assertion:

Students in the treatment group demonstrated a better understanding of the structure of the Unit Circle than students in the comparison group.

The evidentiary warrant (Erikson, 1986) for this assertion, unlike the two previous assertions, is provided primarily by the results of interviews conducted with students. One of the main points of discussion in each interview was the Unit Circle and its organization and structure. Students were given an opportunity to describe how they would determine the values of the common angles and their coordinates on a Unit Circle, and their explanations provide much of the evidence for Assertion 3.

The quiz and test items listed in Table 5 show raw scores that do not clearly favor either the visualization group or the comparison group. This result is not surprising and points out the inherent difficulty in assessing student work samples. Even with a team of expert graders carefully evaluating each item, it is impossible to determine precisely what a student was thinking based solely upon his written work. In some cases, where students gave clear and detailed explanations of their solution processes, reasonable inferences could be made about the degree of understanding possessed by the student. In other cases where minimal written work was provided, this was impossible to assess accurately. Making inferences from written work samples is a difficult task.

Table 5:
Summary of Unit Circle Scores

Item	Group A Comparison	Group B Visualization	Difference
Q2.5	2.82	2.27	-0.55
Q2.9	5.00	4.67	-0.33
T1.18	4.64	4.85	0.21
Q2.10	4.18	4.18	0.00
T1.19	4.49	5.00	0.51
Q2.11	4.00	3.95	-0.05
T1.20	4.74	4.26	-0.48
Q2.14	3.52	3.90	0.38
T1.23	4.49	4.49	0.00
Mean	4.74	4.70	-0.04

The questions about the Unit Circle whose results appear in Table 5 were of several types:

- Given an angle on the Unit Circle, state the coordinates of that point
- Find the angle corresponding to a given ordered pair
- Find values for the six trig functions given a particular common angle
- Evaluate expressions such as $\sin \frac{5\pi}{6} + \cos \frac{5\pi}{4}$

When all of the mean scores on these nine items are examined (see Table 5), it becomes evident that the two groups did not fare much differently. The overall difference between groups on these items was only 0.04 points, with neither group showing more than a half point advantage on any single item. Inspection of student work samples, however, reveals some striking differences between groups, and this was most apparent on the item shown below which appeared as Quiz 2.11 and Test 1.20.

Q2.11 Evaluate the six trig functions for an angle $\theta = \frac{2\pi}{3}$ radians.

T1.20 Evaluate the exact (no decimal approximations) six trig functions for an angle $\theta = \frac{5\pi}{3}$ radians.

One of the instructional goals for this unit was to have students learn the important relationship between coordinates of points on the Unit Circle and the values of the trigonometric ratios. For this reason I looked at student work samples to determine to what extent members of each group used the coordinates of points on the Unit Circle to answer this question. I also looked for evidence of the use of right triangle trigonometry, since there would be no need for this if the student had a deep understanding of the relationship between coordinates and trigonometric ratios.

Here is where a startling difference between groups became apparent. On the quiz item, only one student in the comparison group showed the coordinates of the point on the Unit Circle

corresponding to the angle $\frac{2\pi}{3}$ radians. Six students in the visualization group wrote the

coordinates $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ as part of their solution strategy. When the number of students who used

right triangle trigonometry to solve the problem is considered, the opposite relationship holds. Four out of eleven students in the comparison group drew right triangles on their circles, while only two students in the visualization group used this approach. The solution strategy used by Kate (visualization group) is shown in Figure 9 and illustrates how she sketched a circle to locate the angle and then used the coordinates of that point on the Unit Circle to derive the values of all six trigonometric ratios.

The results are similar on the test item, though not quite as pronounced. Five students in the visualization group used ordered pairs, while only two students in the comparison group did so. The number of students who drew right triangles on the test went down; three students in the comparison group used triangles versus only two in the visualization group.

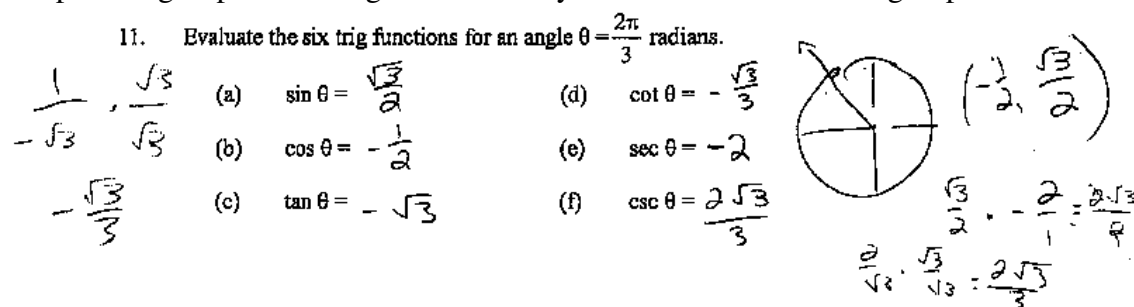


Figure 9. Kate's solution to Q2.11

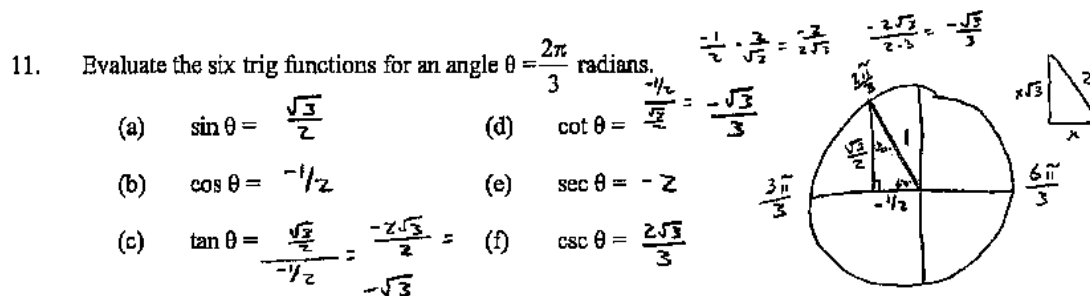


Figure 10. Danielle's solution to Q2.11

More students in the visualization group were able to use coordinates of points on the Unit Circle to obtain values of trigonometric ratios than the students in the comparison group, and they demonstrated less of a tendency to draw and use right triangles as a way to solve the problem. The solution approach used by Danielle (comparison group) in Figure 10 is a good example of the right triangle approach favored by students in Mr. Brownell's class.

The difference between groups in regard to solution strategies can be linked to the types of instructional techniques used by the two teachers. Mr. Brownell showed students several times how angles and coordinates were related, and he made regular use of the relationships that existed within special right triangles. He drew diagrams on the board, sketched reference triangles, and found coordinates for points on the Unit Circle corresponding to common angle values.

Each of his demonstrations and examples, however, took time, because Mr. Brownell had to first draw the circle and then the appropriate triangle before he could begin his solution.

Furthermore, it was difficult for Mr. Brownell to demonstrate the connection that existed between points in the different quadrants having the same reference triangle. As far as his students were concerned, each problem that Mr. Brownell solved was unique and required a specific diagram and strategy to solve.

Stella was able to present solutions more quickly and more accurately by using prepared PowerPoint slides that had been carefully set up to help students see how angles in one quadrant related to angles in the other quadrants. Stella used Sketchpad to help students see how reference angles existed in groups of four, one per quadrant. She was able to help students develop an understanding of the simple relationship between points in one quadrant and the corresponding coordinates of points in other quadrants.

This was accomplished by using PowerPoint slides to demonstrate and then reinforce the relationships again and again, quickly and accurately. Some of the slides included all of the answers for students to look over, and other slides had values missing for students to determine. Stella took full advantage of the technology she had available to paint the best possible picture for her students and help them utilize concepts of symmetry to make a challenging task manageable.

Students who are confident in their ability to determine coordinates of any common angle on the Unit Circle can focus their attention on the relationship between coordinates and trigonometric ratios. Because Stella's students developed this confidence, they were able to focus better on the next step in the process: the relationship between coordinates and trigonometric ratios. They were able to develop an understanding and appreciation for the manner in which coordinates of points on the Unit Circle allow you to easily determine the values of the six trigonometric ratios.

One other source of information was examined to see whether it revealed any differences between groups with regard to the Unit Circle: the interview conducted with students following Test 1. An important part of the interview was a discussion of the Unit Circle, its organization, and importance in the study of trigonometry. Each member of the project team devoted a portion of the 15-minute interview to this topic. Students were asked to explain how the Unit Circle was organized and show or describe how they would go about filling in the values on a blank Unit Circle. These interviews revealed differences between groups and among students that was not apparent based only upon the written evidence of quizzes and tests.

Students in the comparison group showed a strong preference toward the use of a memorized pattern of values that appeared in the numerators and denominators of the fractional coordinates of points on the Unit Circle. This involved counting "1, 2, 3" for the denominators and then "3, 2, 1" for the numerators, and inserting square root symbols in the numerators, a process that evidenced no understanding of the relationship between those coordinates and their associated angles.

This technique was similar to one used by students in the visualization class to recall the measures of common angles in each quadrant. Consecutive differences between angles were "30, 15, 15, and 30" degrees, and many students used this technique to identify the sizes of common angles in each quadrant.

One student in the comparison group who struggled to explain the Unit Circle was Molly. I interviewed Molly and heard her use the same counting strategy to determine the coordinates of points on the Unit Circle. I asked Molly if she could explain the coordinates any other way without using the pattern, and she said, "Definitely. I couldn't tell you, but I know that's what they are." Since Molly could not make any headway on her own, I tried to get her started by sketching a right triangle in the first quadrant for the

angle $\frac{\pi}{6}$. Molly remembered that it had something to do with the special right triangles. I asked, “Is there anything special about that triangle that you may have learned somewhere along the way?” Molly replied, “There are those rules where this is half, just something that I should have memorized and I didn’t.”

Molly’s reference to rules and memorization is telling. A senior, it is undoubtedly true that Molly’s notion of mathematics as a series of rules, formulas, and memorization has been nurtured and developed over many years of formal mathematics, not just during her time in precalculus. What is also clear, however, is that nothing had occurred during this unit on trigonometry to help Molly make sense of the underlying relationships that gave the Unit Circle meaning.

Interviews with Megan, Miranda, and Samantha (all in the visualization group) showed a strategy of finding first quadrant values and using them to obtain values in other quadrants. Megan referred to “flipping over” the axes to get her values in other quadrants, and Miranda mentioned that there are four congruent triangles, one in each quadrant, having the same pairs of coordinates except for their signs.

Glenda (visualization group) gave a thorough description of how she would reconstruct the Unit Circle values and stated that she would use the symmetrical relationships among quadrants as well. She also described another relationship that did not come up in any of the other interviews. In the first quadrant Glenda noted that the reference triangle for a 30 degree angle is the same as the reference triangle for a 60 degree angle differing only in orientation. Since the two triangles are the same, their leg lengths are the same only reversed, according to Glenda, so she knew that the 30- and 60-degree angles had the same pair of coordinates differing only by their order.

Interviews were conducted with 8 of the 13 students in the study who were members of the visualization group. All eight of the students indicated in their interviews that there were symmetrical relationships that helped them determine angles and coordinates for common angle points on the Unit Circle. Furthermore, not one of the students in the visualization group made any reference to the type of rote counting technique used by students in the comparison group for remembering coordinate numerators or denominators. Nor did any of Stella’s students make reference to the sequence of differences between consecutive angles on the Unit Circle (30-15-15-30) that students in Mr. Brownell’s class made.

Most of Stella’s instruction on the Unit Circle relied on the use of a series of PowerPoint slides that had been prepared for her. The slides were set up so that Stella could pose a question for the class, have them predict the results, and then immediately confirm the results. She had the angles and coordinates organized in such a way that they were presented to the students in a logical sequence based on multiples and symmetry of points on the Unit Circle. Stella used this series of slides on several occasions as a way of helping students develop an understanding of the relationships that exist in the Unit Circle that make its angles and coordinates relatively easy to determine without memorization.

During one of the early lessons on the Unit Circle, one of Stella’s students made an announcement to the class:

Allison: I have a revelation to make.

Stella: A revelation?

Allison: Yes. The next point over from the 90 degrees is going to be the same as the 30 degrees.

Stella moves over to the SmartBoard and points toward an angle.

Stella: Are you talking about this point (pointing toward the location in the first quadrant where an angle of 60 degrees would be located)? What are you guessing that this point will be?

Allison: It's going to be the same coordinates as the 30 degree angle—they're just going to be flipped.

Stella: They're going to be flipped? How do you know that?

Allison: Because the radius is the same—the hypotenuse is the same—it's the same triangle—it's just flipped.

Stella adds to Allison's explanation to make sure that everyone understands the important point being made.

Stella: It's just that 30 degrees is up here (pointing to the angle whose vertex is on the circle) and now the 60 degrees is down here (indicating the angle whose vertex is at the origin).

Stella gave students a chance for Allison's keen observation to sink in and then took the opportunity to use her "revelation" to make another point. She asked students to tell her how long the two legs of the triangle are, starting with the shorter leg. Once students recognized that the short horizontal segment must be one half, Stella made sure the students knew which coordinate that had to be. She did the same for the y-coordinate and then asked Allison to tell the class what the ordered pair had to be. Allison proudly announced that the ordered pair

was $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The end result of this discussion was that students had the coordinates for a point

located at 60 degrees based entirely upon their prior knowledge about the 30 degree angle, thanks to Allison's revelation.

Mr. Brownell used the white board to draw sketches of common angles in various quadrants as part of his instruction dealing with the Unit Circle. His strategy and approach was consistent: locate the point on the Unit Circle associated with the given angle and then draw a reference triangle with its right angle on the x-axis. Then use the 30-60-90 or 45-45-90 right triangle relationships to determine the lengths of each leg of the triangle. Once this was accomplished, students were able to use the lengths of the legs as the x- and y-coordinates of the given point on the Unit Circle.

Mr. Brownell also discussed with the class the fact that points in one quadrant corresponded to points in the other quadrants, but he was not able to do so dynamically, given the limited resources at his disposal. Whereas Stella was able to quickly click from 30 to 150 to 210 and then to 330 degrees in sequence to show how all four points had the same coordinates (except for sign), Mr. Brownell could only draw static diagrams to make the same teaching point, and the time required for him to draw each triangle reduced the number of such examples he was able to provide.

The instruction Stella and Mr. Brownell provided to their students with regard to the structure of the Unit Circle was intended to help students gain a better understanding of how the Unit Circle was organized. This in turn would allow students to determine values of the angles, coordinates, and trigonometric ratios for any points on the Unit Circle, an important ability that would help students with their transition to the graphs of the trigonometric functions to follow.

Results indicate that the students in Stella's class were able to obtain a better sense of the structure of the Unit Circle than students in Mr. Brownell's class. Stella's consistent use of visual

images to emphasize the symmetry of the Unit Circle was an important factor which contributed to her students' success.

Procedural Competence and Conceptual Understanding

There has been some concern that the use of technology may lead to a loss of procedural competence, so the data was analyzed to see if such a trend existed in this study. Items which were considered to be of a procedural nature were examined by group to see if the group that received traditional instruction outperformed the visualization group. With one exception, this was found not to be the case and led to the assertion that students whose instruction includes a significant focus upon the use of technology for visualization do not demonstrate any loss of procedural competence with the exception of items dealing with evaluation of trigonometric ratios for specific angles.

Trigonometric Misconceptions

Finally, since there was no research that investigated the types of misconceptions that students demonstrate when learning about these topics of trigonometry, the data was carefully analyzed to look for persistent problems that students had with the concepts of radians, reference angles, and the Unit Circle. A number of such misconceptions was identified and classified as part of this study; a full discussion of these trigonometric misconceptions will be presented in a subsequent article.

Educational and Scientific Importance of the Study

These findings have implications for teacher educators who prepare future mathematics teachers and also for school systems as they integrate technology into mathematics classrooms. Teacher preparation programs must ensure that preservice teachers are able to make appropriate use of current technology in their classroom as an important tool for visualization. Preservice teachers must be provided an opportunity to learn how to use this technology in methods classes and then allowed to work in a student teaching situation where such technology is available for their use.

It should also be stated that a network of support is extremely important for preservice teachers. They need technical and pedagogical support to take advantage of technology. Stella's student teaching experience might have been much different had she not had the members of the project team available to develop some of the tools that she used with her classes. Not only did the team create many of the files that Stella used with her classes, Stella was also often allowed to practice using the files in mock teaching episodes with her fellow preservice teachers during a methods seminar. These practice sessions enabled the research team to help Stella develop a sense for how to make best use of the technological tools with her class. Inservice teachers must also receive the support needed to make the transition from a traditional delivery system to one that incorporates technology for visualization into their teaching.

This study is an important addition to the body of knowledge dealing with representations and technology use in the mathematics classroom because its focus is on student conceptual understanding. The findings show that appropriate use of technology for visualization can lead to improved student understanding without a loss of procedural competence.

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